Analysis 3: Exercises 6

1. Is it true on a set of infinite measure that convergence a.e. implies convergence in measure? Justify your answer.

2. A sequence of measurable functions $\{f_n\}$ is said to be fundamental in meaure if for any $\epsilon > 0$

$$\lim_{m,n\to\infty} \mu\{x : \ |f_n(x) - f_m(x)| > \epsilon\} = 0.$$

Show that if $\{f_n\}$ is fundamental in measure then there exists a measurable function f such that $f_n \to f$ in measure.

Show that f_n converges to f in measure then f_n is fundamental in measure.

3. Let (f_n), (g_n) be $\mathbb{R}$-valued $\mathfrak{A}$-measurable functions which converge in measure to $\mathfrak{A}$-measurable functions f and g respectively. Prove that

a) $(a\, f_n + b\, g_n)$ converges in measure to $a\, f + b\, g$ for any $a, b \in \mathbb{R}$;

b) $(|f_n|)$ converges to $|f|$ in measure.

4. Show that if $\mu(\Omega) < +\infty$ and $f_n \to f$ and $g_n \to g$ in measure, then $f_n\, g_n \to f\, g$ in measure.

5. Prove the following versions of Fatou's Lemma. Assume $f_n,\ f \in M_+$ $(n \in \mathbb{N})$.

i) If $f_n \to f$ μ-a.e. then $\int f\, d\mu \leq \underline{\lim} \int f_n\, d\mu$.

ii) If $f_n \to f$ in measure then $\int f\, d\mu \leq \underline{\lim} \int f_n\, d\mu$.

Here, the notation $\underline{\lim}$ stands for the limit inferior $\liminf$.

6. Show that Lebesgue's Dominated Convergence Theorem holds if almost everywhere convergence is replaced by convergence in measure.

7. Let $(\Omega, \mathfrak{A}, \mu)$ be a finite measure space. If f is an $\mathfrak{A}$-measurable function, put

$$r(f) = \int \frac{|f|}{1 + |f|}\, d\mu.$$

Show that a sequence (f_n) of $\mathfrak{A}$-measurable $\mathbb{R}$-valued functions converges in measure to f if and only if $r(f_n - f) \to 0$.

8. If a sequence (f_n) converges in L^p to a function f and a subsequence of (f_n) converges in L^p to g, then $f = g$ a.e.

9. Find an example which shows that convergence in L^p does not imply a.e. convergence.

10. Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a **probability space** (i.e. $\mathbb{P}(\Omega) = 1$). Let $X : \Omega \to \mathbb{R}$ be an $\mathfrak{A}$-measurable **random variable**. Assume that $X \in L^2(\Omega, \mathfrak{A}, \mathbb{P})$. Define

$$\mu = \mathbb{E}[X] = \int X\, d\mathbb{P} \qquad \text{(\textbf{expectation} of } X\text{)},$$

$$\sigma^2 = \mathbb{E}[(X-\mu)^2] \qquad (\textbf{variance} \text{ of } X)$$

Prove **Chebychev's inequality**,

$$\mathbb{P}[|X-\mu| \geq \alpha] \leq (\sigma/\alpha)^2$$

for any $\alpha > 0$.