① <u>Lecture</u>

Recall ν is a charge if $\nu : \underline{X} \to \mathbb{R}$

1) $\nu(\emptyset) = 0$
2) $\nu(\bigcup_n A_n) = \sum_n \nu(A_n)$ A_n are disjoint

Typical examples:

$\nu(A) = \mu_1(A) - \mu_2(A)$ μ_1 & μ_2 are measures

$\nu(A) = \int_A f \, d\mu$

Def Let $\nu : X \to \mathbb{R}$ be a charge

A set $P \in \underline{X}$ is positive wrt ν if $\nu(A \cap P) \geq 0$ $\forall A \in \underline{X}$

A set $N \in \underline{X}$ is negative wrt ν if $\nu(A \cap N) \leq 0$ $\forall A \in \underline{X}$

A set $M \in \underline{X}$ is a null set wrt ν if
$\nu(A \cap M) = 0$ $\forall A \in \underline{X}$

Lemma 1) A measurable subset of a positive set is positive

2) A union of two positive sets is positive

(same true ~~for~~ negative)

Proof 1) Let P be a positive set & $E \subset P$ is measurable $\Rightarrow$

$\nu(E \cap A) = \nu((P \cap E) \cap A) = \nu(P \cap (E \cap A)) \geq 0$ $\forall A \in \underline{X}$

2) P_1 & P_2 are positive

$\nu((P_1 \cup P_2) \cap A) = \nu((P_1 \cap A) \cup (P_2 \cap A))$

or

$\underbrace{P_1 \cap P_2^c}_{\text{positive}} \cup P_2$

② <u>Theorem</u> (Hahn decomposition)

Let ν be a signed measure on $(X, \underline{X})$ then $\exists$ a negative set A^- & positive set $A^+ = X \setminus A^-$

<u>Proof</u> Let us define $\mathcal{N}$ - a set of all negative sets.

$$\beta = \inf_{B \in \mathcal{N}} \nu(B)$$

We can find a sequence $B_n \in \mathcal{N}$ with $B_n \subset B_{n+1}$ and $\lim_{n\to\infty} \nu(B_n) = \beta$. and note that

We define $N = \bigcup_{n\geq 1} B_n$

$$\nu(N) = \lim_{n\to\infty} \nu(B_n)$$

[It is clear that $\nu(N) \leq \nu(B_n) \Rightarrow$
$\Rightarrow \nu(N) \leq \lim \nu(B_n) = \beta$ and since $N \in \mathcal{N}$ we have $\nu(N) \geq \beta$]

We want to show $P = X \setminus N$ is positive. If $E \in \mathcal{N}$

Assume not $\Rightarrow \exists E : \nu(E) < 0$. $\nu(N \cup E) = \nu(N) + \nu(E) < \beta$

Then $N \cup E \in \mathcal{N}$ &
$\Rightarrow$ contradiction

So we know that E is not negative so $\exists C \subset E$ such that $\nu(C) > 0$.

Proceed as follows

1) find $k_1 > 0$ such that $\exists E_1 \subset E$ with $\nu(E_1) \geq k_1$ and k_1 is the maximal of such numbers

$$\left[k_1 = \sup_{k>0} \{ \exists A \subset E \text{ with } \nu(A) \geq k \} \right]$$

2) Consider $E \setminus E_1$ $\left(\nu(E \setminus E_1) = \nu(E) - \nu(E_1) < 0 \right)$
and find $k_2 > 0$ such that $\exists E_2 \subset E \setminus E_1$
$\nu(E_2) \geq k_2$ and k_2 is maximal

$$k_2 = \sup_{k>0} \{ \exists A \subset E \setminus E_1 \text{ with } \nu(A) \geq k \}$$

It is clear $k_1 \geq k_2$

We continue this procedure and find a sequence of disjoint sets $\{E_i\} \subset E$ with $\nu(E_i) \geq k_i$

This is an infinite sequence, otherwise we find a larger negative set.

Clearly $k_i \to 0$ as otherwise

$$\nu\Big(\bigcup_i E_i\Big) = \infty$$

We define $F = E \setminus \bigcup_{i=1}^{\infty} E_i$.

Clearly $\nu(F) < 0$ and if F is negative $\Rightarrow$ we have a contradiction.

So $\exists\, A \subset F$ such that $\nu(A) > 0$

Let $k_N > 0$ be such that $\nu(A) \geq k_N + \varepsilon$

Since $A \subset F \subset E \setminus \bigcup_{i=1}^{N-1} E_i$ & we have

$$k_N = \sup_{k>0} \Big\{ \exists\, A \subset E \setminus \bigcup_{i=1}^{N-1} E_i \text{ with } \nu(A) \geq k \Big\}$$

we actually have a contradiction.

<u>Lemma</u> let P_1, N_1 & P_2, N_2 be Hahn decompositions for ν.

Then $\nu(A \cap P_1) = \nu(A \cap P_2)$ & $\nu(A \cap N_1) = \nu(A \cap N_2)$ $\quad \forall A \in \underline{\mathfrak{X}}$

<u>Proof</u> $\nu(A \cap (P_1 \cup P_2)) = \nu(A \cap P_1) + \nu(A \cap (P_2 \setminus P_1)) =$

$= \nu(A \cap P_2) + \nu(A \cap (P_1 \setminus P_2))$

$P_2 \setminus P_1 \subset N_1$ & $P_1 \setminus P_2 \subset N_2$ $\Rightarrow$ $\nu(A \cap (P_2 \setminus P_1)) = 0$

$\nu(A \cap (P_1 \setminus P_2)) = 0$

$P_2 \setminus P_1 = (P_2 \cap P_1^c) \cap A = P_2 \cap (P_1^c \cup A)$

$\Rightarrow \quad \nu(A \cap P_1) = \nu(A \cap P_2)$.

(7)

Linear functionals & duality

<u>Def</u> A linear functional on $L^p(X)$ is a map $G: L_p \to \mathbb{R}$ such that $\forall a, b \in \mathbb{R}$ & $\forall f, g \in L^p$

$$G(af + bg) = aG(f) + bG(g)$$

A linear functional is bounded if $\exists M \in \mathbb{R}$ such that $\forall f \in L_p$

$$|G(f)| \le M \|f\|_p$$

The norm of G is $\|G\| \equiv \sup\{|G(f)| : f \in L_p, \|f\|_p \le 1\}$

<u>Theorem</u> A collection of all bounded linear functionals B^* on L_p (or any Banach space) is a Banach space.

<u>Proof</u>
1) B^* is a vector space
2) $\|\cdot\|$ is a norm on B^*
3) Completeness

Let $\{F_n\}$ be Cauchy sequence, i.e. $\|F_n - F_m\|_* \to 0$

Hence $|F_n(f) - F_m(f)| \to 0 \quad \forall f \in L_p$

$\Rightarrow \{F_n(f)\}$ is Cauchy in $\mathbb{R}$ $\Rightarrow$

$\Rightarrow F_n(f) \to a \equiv F(f)$

It's clear F is a linear functional

Moreover $|F(f)| = \lim\limits_{n\to\infty} |F_n(f)| \le M\|f\|$

As $\|F_n\| \le M$

So F is bounded linear functional

$\forall f : \|f\|_p \le 1$ we have

$$|F_n(f) - F(f)| = \lim_{m\to\infty} |F_n(f) - F_m(f)| \le \lim_{m\to\infty} \|F_n - F_m\|$$

$\Rightarrow \|F_n - F\| \to 0$ ∎

② **Theorem** ① Let $\infty > q > 1$ & $p = \frac{q}{q-1}$, $g \in L_q$

Then $G : L_p \to \mathbb{R}$ defined as

$G(f) = \int_X f g \, d\mu$ is a bounded

linear functional and $\|G\| = \|g\|_q$

② Let $q = 1$ and $p = \infty$, $g \in L_1$

Then $G : L_\infty \to \mathbb{R}$

$G(f) = \int_X f g \, d\mu$ is a bounded

linear functional & $\|G\| = \|g\|_1$

③ Let $q = \infty$, $p = 1$, $g \in L_\infty$

$G : L_1 \to \mathbb{R}$ $G(f) = \int_X f g \, d\mu$

is a bounded linear functional

& $\|G\| = \|g\|_\infty$

Proof 1) G is linear $|G(f)| \le \|g\|_q \|f\|_p$

$\Rightarrow$ $\|G\| \le \|g\|_q$.

take $h(x) = \operatorname{sgn}(g(x)) \, |g(x)|^{q-1}$

Clearly $h \in L^p$, $\|h\|_p = \|g\|_q^{q-1}$

$G(h) = \int_X h g \, d\mu = \int_X |g|^q d\mu = \|g\|_q^q =$

$= \|g\|_q \|h\|_p$

$\Rightarrow G\left(\frac{h}{\|h\|_p}\right) = \|g\|_q$ ■

② G is linear $|G(f)| \le \|f\|_\infty \|g\|_1$

$\Rightarrow$ $\|G\| \le \|g\|_1$

Take $h(x) = \operatorname{sgn}(g(x))$

$G(h) = \int_X |g| \, d\mu = \|g\|_1$ ■

Theorem (Riesz Representation)

Let (X, μ) be a measure space and μ is σ-finite measure.
Let G be a bounded linear functional on $L_p(X,\mu)$ $1<p<\infty$. Then $\exists\, g \in L_q$ $\left(q = \frac{p}{p-1}\right)$ such that

$$G(f) = \int_X f g \, d\mu \quad \forall f \in L_p$$

and $\|G\| = \|g\|_q$

Proof Take any $G \in L_p^*$ and define

$$\nu(A) = G(\chi_A) \quad \forall A \in \underline{X} \quad \left(\text{we assume } \mu(X) < \infty\right)$$

We can check that ν is a charge

1) $\nu(\phi) = G(0) = 0$

2) Take $\{A_n\}_{n=1}^{\infty}$ disjoint

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = G\left(\chi_{\bigcup_{n=1}^{\infty} A_n}\right) = G\left(\sum_{n=1}^{\infty} \chi_{A_n}\right) =$$

$$\textcircled{=} \lim_{N\to\infty} G\left(\sum_{n=1}^{N} \chi_{A_n}\right) = \lim_{N\to\infty} \nu\left(\bigcup_{n=1}^{N} A_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \nu(A_n)$$

$$\sum_{n=1}^{N} \chi_{A_n} \to \sum_{n=1}^{\infty} \chi_{A_n} \text{ in } L^p \Rightarrow G\left(\sum_{n=1}^{N} \chi_{A_n}\right) \to G\left(\sum_{n=1}^{\infty} \chi_{A_n}\right)$$

Since ν is a charge we know also that

$$\nu \ll \mu \quad \left(\mu(A) = 0 \to \chi_A = 0 \text{ a.e.} \Rightarrow \right.$$

$$\left. \Rightarrow \nu(A) = F(\chi_A) = 0 \right)$$

$$\left(\|F(\chi_A)\| \le C \|\chi_A\| = 0 \right)$$

(4) By Radon–Nikodym we have

$\nu(A) = \int_A g \, d\mu$ for some $g \in L$.

We have to show:

1) $g \in L^q$ 2) $G(f) = \int_X f g \, d\mu \quad \forall f \in L_p$

For any simple $\varphi(x)$ we have

$$G(\varphi) = \int_X \varphi g \, d\mu$$

$$\sup \{ \int_X \varphi g \, d\mu , \varphi \in L^p(X), \|\varphi\|_p \leq 1 \ \varphi \text{ is simple} \}$$

$\leq \|G\|_*$ (as we take sup over smaller set)

On the other hand

$$\|g\|_q = \sup \{ \int_X f g \, d\mu , \|f\|_p \leq 1 \} =$$

$$= \sup \{ \int_X \varphi g \, d\mu , \|\varphi\|_p \leq 1 , \varphi \text{ is simple} \}$$

Hence $g \in L_q$

2) Take any $f \in L^p$ $\exists \varphi_n \xrightarrow{L_p} f$ $\varphi_n \in L_p$, φ_n – simple

$F(\varphi_n) = \int_X g \varphi_n \, d\mu$

$F(\varphi_n) \to F(f)$ and $\int_X g \varphi_n \to \int_X g f$ $\Rightarrow$ done.

Def ① Let μ^* be an outer measure on $P(X)$. We say that $E \subset X$ is μ^*-measurable & $\in \mathcal{A}^*$ if
$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$
$\forall A \in P(X)$.

Theorem 1) $\mathcal{A}^*$ is a σ-algebra containing $\mathcal{A}$

2) μ^* is σ-additive on $\mathcal{A}^*$.

Proof

1) $X, \phi \in \mathcal{A}^*$

2) if $E \in \mathcal{A}^* \Rightarrow E^c \in \mathcal{A}^*$ (by def)

3) $E, F \in \mathcal{A}^* \Rightarrow E \cap F \in \mathcal{A}^*$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$
$$\mu^*(A \cap E) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) +$$
$$\mu^*(A) = \mu^*(A \cap (E \cap F)) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c)$$

But
$$\mu^*(A \cap (E \cap F)^c) = \mu^*(A \cap (E \cap F)^c \cap E) + \mu^*(A \cap (E \cap F)^c \cap E^c) =$$
$$= \mu^*(A \cap F^c \cap E) + \mu^*(A \cap E^c)$$

Hence $\mu^*(A) = \mu^*(A \cap (E \cap F)) + \mu^*(A \cap (E \cap F)^c)$

We see that $\mathcal{A}^*$ is an algebra.

(2) ~~We can also show~~ if $E, F \in \mathcal{A}^*$, $E \cap F = \phi \Rightarrow$

$$\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F).$$

$$\mu^*(A \cap (E \cup F)) = \mu^*(A \cap (E \cup F) \cap E) +$$

$$+ \mu^*(A \cap (E \cup F) \cap E^c) =$$

$$= \mu^*(A \cap E) + \mu^*(A \cap F).$$

So μ^* is additive on $\mathcal{A}^*$.

We have to show $\mathcal{A}^*$ is σ-algebra

Take $E = \bigcup_{k=1}^{\infty} E_k$, $E_k \in \mathcal{A}^*$, disjoint

Define $F_n = \bigcup_{k=1}^{n} E_k$, we know $F_n \in \mathcal{A}^*$

Take any $A \subset X$:

$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ by semi-additivity.

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) =$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap F_n^c) \geq$$

$$\geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c)$$

Take $\lim n \to \infty$

$$\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \geq$$

$$\geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Hence $E \in \mathcal{A}^* \Rightarrow \mathcal{A}^*$ is σ-algebra and μ^* is countably additive.

(3) $\mathcal{A} \subset \mathcal{A}^*$

Take $E \subset \mathcal{A}$ & $A \subset X$

$\mu^*(A) \leq \mu^*(E \cap A) + \mu^*(A \cap E^c)$ by semi-additivity

Fix $\varepsilon > 0$ & find $F_n \in \mathcal{A}$ such that $A \subset \bigcup_{n=1}^{\infty} F_n$

and $\mu^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} \mu^*(F_n)$

③

Since $E, F_n \in \mathcal{A}$ and $\mu^* \equiv \mu$ on $\mathcal{A}$

we have

$$\mu^*(F_n) = \mu^*(F_n \cap E) + \mu^*(F_n \cap E^c)$$

$$\mu^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} \mu^*(F_n \cap E) + \mu^*(F_n \cap E^c) \geq$$

$$\geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Send $\varepsilon \to 0$ and

<u>Note</u> $(X, \mathcal{A}^*, \mu^*)$ is a complete measure space

Let $N \in \mathcal{A}^*$, $\mu^*(N) = 0$ take $E \subset N$

$$\mu^*(A) = \mu^*(A \cap N) + \mu^*(A \cap N^c)$$

We have to show

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\mu^*(A \cap E) = 0 \quad \text{as} \quad A \cap E \subset N$$

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \text{by subadd.}$$

$$\mu^*(A) \geq \mu^*(A \cap E^c) + \underbrace{\mu^*(A \cap E)}_{=0}$$

$\Rightarrow$ done.

(1) Equivalence of definitions of measurable sets.

Theorem (Hahn uniqueness Th-m)
Let μ be a σ-finite premeasure on $\mathcal{A}$. Then
extension of μ to a measure on $\mathcal{A}^*$ is unique.

Proof We first prove it for finite measures.
Assume $\exists\, \nu$ such that ν is a measure on $\mathcal{A}^*$ and

$$\nu|_{\mathcal{A}} \equiv \mu|_{\mathcal{A}}$$

We have to show that $\nu|_{\mathcal{A}^*} = \mu|_{\mathcal{A}^*}$

Take $E \in \mathcal{A}^*$ $\forall \varepsilon > 0$ $\exists\, E_n \in \mathcal{A}$ and
such that $E \subseteq \bigcup_{n\geq 1}^{\infty} E_n$

$$\mu^*(E) \geq \sum_{n\geq 1}^{\infty} \mu^*(E_n) - \varepsilon$$

Since ν is a measure on $\mathcal{A}^*$ we have

$$\nu(E) \leq \sum_{n\geq 1}^{\infty} \nu(E_n) = \sum_{n\geq 1}^{\infty} \mu(E_n) \leq \mu^*(E) + \varepsilon$$

Hence $\nu(E) \leq \mu^*(E)$
We also have $\nu(E^c) \leq \mu^*(E^c)$

$$\mu^*(X) = \mu^*(E) + \mu^*(E^c) \geq \nu(E) + \nu(E^c) = \nu(X)$$

Hence $\mu^*(E) = \nu(E)$ $\forall E \in \mathcal{A}^*$

Extension to σ-finite case is a HW.

(2)

Def $\mathbb{F}^*$ is the σ-algebra of Lebesgue measurable sets
and ℓ^* is the Lebesgue measure.

Def $\mathbb{B}$ is the smallest σ-algebra containing $\mathbb{A}$ and hence
$\mathbb{B} \subset \mathbb{F}^*$
$\mu^\circ|_{\mathcal{B}}$ is called Borel measure.
It is not complete.