MTI Exercises 3: Solutions

1. Use Chebyshev's inequality.

2. Was shown in lectures. Use Theorem 4.14 from the notes for $|f|$.

3. Use previous result.

4. For $f \geq 0$ by Lemma 2.15 we have a monotone sequence of simple functions $\phi_n \to f$ and $\phi_1 \leq ... \leq \phi_n \leq \phi_{n+1} \leq ... \leq f$. Using MCT we have that for any $\epsilon > 0$ there exists N such that for all $n > N$
$$|\int \phi_n d\mu - \int f d\mu| < \epsilon.$$
Splitting general f into a difference of f_+ and f_- gives the result.

5. A variation of the problem considered in lectures.

6. Fix $\epsilon > 0$ and choose N such that for all $n \geq N$ we have that $\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon/\mu(X)$. We have that for $n \geq N$
$$\int f_n \mathrm{d}\mu \leq \int (f + \epsilon/\mu(X)) \mathrm{d}\mu$$
which means that since $\int \epsilon/\mu(X) \mathrm{d}\mu(x) = \epsilon$
$$\int f_n \mathrm{d}\mu \leq \int f \mathrm{d}\mu + \epsilon.$$
We also have
$$\int f \mathrm{d}\mu \leq \int (f_n + \epsilon/\mu(X)) \mathrm{d}\mu$$
which means that
$$\int f_n \mathrm{d}\mu \geq \int f \mathrm{d}\mu - \epsilon.$$
The result follows.

   If $\mu(\Omega) = \infty$ then take $f_n = 1/n$ on the interval $[n, 2n]$ and 0 outside, and see the contradiction.

7. We use two results here – MCT and DCT. It's trivial to show that $\lim_{N\to\infty} \sum_{n=1}^{N} |f_n| = \sum_{n=1}^{\infty} |f_n| \equiv g \in M^+$. Using MCT we know that
$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} |f_n| d\mu = \int_X g d\mu.$$
Therefore $g \in L$. Now since $\sum_{n=1}^{\infty} |f_n| \to g \in L$ we have $|\sum_n f_n| \leq g < \infty$ a.e. (since $g \in L$)and hence $\sum_n f_n = f \in L$. Using DCT we obtain the result.

8. Trivial as $\left|\int_X |f| d\mu - \int_X |g| d\mu\right| \leq \int_X |f - g| d\mu$.