MTI Exercises 7: Solutions

1. Since $A \subset B \cup (A\Delta B)$ and $B \subset A \cup (A\Delta B)$ we have
$$\mu^*(A) \le \mu^*(B) + \mu^*(A\Delta B), \quad \mu^*(B) \le \mu^*(A) + \mu^*(A\Delta B).$$
This implies $|\mu^*(A) - \mu^*(B)| \le \mu^*(A\Delta B)$.

2. Let E be measurable according to the definition above. Take any $\epsilon > 0$ then there exists a set $A \in \mathbb{A}$ such that $\mu^*(A\Delta E) < \epsilon$. Since
$$A\Delta E = A^c \Delta E^c$$
we have $\mu^*(A^c\Delta E^c) < \epsilon$, where $A^c \in \mathbb{A}$. Using result from question 1, for any $B \subset X$ we obtain
$$|\mu^*(B \cap A) - \mu^*(E \cap B)| \le \mu^*((B \cap A)\Delta(E \cap B)) \le \mu^*(A\Delta E) < \epsilon,$$
$$|\mu^*(B \cap A^c) - \mu^*(E^c \cap B)| \le \mu^*((B \cap A^c)\Delta(E^c \cap B)) \le \mu^*(A^c\Delta E^c) < \epsilon.$$
From the above inequalities we have
$$\mu^*(B \cap E) + \mu^*(E^c \cap B) \le \mu^*(A \cap B) + \mu^*(A^c \cap B) + 2\epsilon. \quad (1)$$
However since $A \in \mathbb{A}$ the following is true:
$$\mu^*(A \cap B) + \mu^*(A^c \cap B) = \mu^*(B).$$
Therefore taking $\epsilon \to 0$ we obtain
$$\mu^*(B \cap E) + \mu^*(E^c \cap B) \le \mu^*(B).$$
The other inequality follows from semi-continuity of μ^*.

The second part is done in Warwick lecture notes, page 14.

3. Since E is measurable we have
$$\mu^*(E \cup A) = \mu^*(E) + \mu^*(A \cap E^c)$$
and
$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$
The result follows.

4. Let I_n be any sequence of sets from $\mathbb{F}$ covering A then $I_n + a$ cover $A + a$. By semi-additivity of l^* we have
$$l^*(A + a) \le \sum_n l(I_n + a) = \sum_n l(I_n).$$
Since it's true for any sequence I_n by definition of l^* we obtain
$$l^*(A + a) \le l^*(A).$$
In the same way we can prove that $l^*(A) \le l^*(A + a)$.

5. Assuming that Vitali set is measurable and recalling translation invariance of l^* we obtain a contradiction in the same way as in the proof of l^* being not sigma additive.

6. If μ is σ-finite then we can represent $\mathbb{R}$ as a countable union of sets of finite measure. As $\mathbb{R}$ is uncountable then at last one of these sets have to be uncountable and so it will have infinite measure, hence μ is not σ-additive.

7. Use the definition of l^* and note that $\beta(\mathbb{F}) = \mathbb{B}$.

8. This is standard.