MT&I: Exercises 8

1. Let μ^* be an outer measure on $P(X)$. Show that for any $A, B \subset X$ the following inequality holds

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A\Delta B) = \mu^*(A \setminus B) + \mu^*(B \setminus A)$$

2. Show that the following definition of measurability is equivalent to the one given in lectures:
A set $E \subset X$ is measureable if for any $\epsilon > 0$ there exists a set $A \in \mathbb{A}$ such that

$$\mu^*(A\Delta E) < \epsilon.$$

3. Let μ^* be an outer measure and $E \subset X$ be a measurable set. Then for any $A \subset X$ we have

$$\mu^*(E) + \mu^*(A) = \mu^*(E \cap A) + \mu^*(E \cup A).$$

4. Prove that l^* is translation invariant, i.e. for any bounded $A \subset \mathbb{R}$ we have $l^*(A) = l^*(a + A)$ for all $a \in \mathbb{R}$.

5. Show that Vitali set is non-measurable.

6. Let μ be a counting measure on $P(\mathbb{R})$. Show that it is not σ-finite.

7. Let $A \in \mathbb{F}^*$ be a Lebesgue measurable subset of $\mathbb{R}$. Show the following.

i) There exists $B \in \mathfrak{B}$ such that $A \subseteq B$ and $l^*(B \setminus A) = 0$.

ii) There exists $B \in \mathfrak{B}$ and $N \in \mathbb{F}^*$ such that $A = B \cup N$ and $l^*(A) = l^*(B)$ and $l^*(N) = 0$.

8. Extend Hahn uniqueness theorem for σ-finite measures.